

# THE PRINCIPLE OF STATIONARY ACTION IN THE CALCULUS OF VARIATIONS

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**ABSTRACT.** We review some techniques from non-linear analysis in order to investigate critical paths for the action functional in the calculus of variations applied to physics. Our main intention in this regard is to expose precise mathematical conditions for critical paths to be minimum solutions in a variety of situations of interest in Physics. Our claim is that, with a few elementary techniques, a systematic analysis (including the domain for which critical points are genuine minima) of non-trivial models is possible. We present specific models arising in modern physical theories in order to make clear the ideas here exposed.

## 1. INTRODUCTION

The calculus of variations is one of the oldest techniques of differential calculus. Ever since its creation by Johann and Jakob Bernoulli in 1696-97, to solve the problem of the brachistochrone (others solved it, too: Newton, Leibniz, Tschirnhaus and L'Hôpital, but their methods were different), it has been applied to a variety of problems both in pure and applied mathematics. While occupying a central place in modern engineering techniques (mainly in control theory, see [9], [38], [47], [65], [75]), it is in physics where its use has been promoted to the highest level, that of the basic principle to obtain the equations of motion, both in the dynamics of particles and fields: the principle of stationary action (see [5], [10], [32], [56]). Accordingly to that point of view, almost every text on mechanics include a chapter on the calculus of variations although, surprisingly enough, the treatment in these texts is expeditious and superficial, directly oriented towards the obtention of Euler-Lagrange's equations, leaving aside the question of whether the solutions are true minima or maxima, despite the importance of this distinction (for instance, while in fields such as optics one is interested in the minimal optical length, in stochastic dynamics one seeks to maximize the path entropy [71]. On the other hand, while the principle of stationary action just selects critical paths, experimentally an actual minimum is detected in some systems, see [25]).

It is interesting to note that the principle was once called the principle of least action, although it was soon realized that many physical phenomena does not follow a trajectory that realizes a minimum of the action, but just a critical path (the main example is the harmonic oscillator, whose trajectory in phase space only minimizes the action for a time interval of length which depends on its frequency, see Sec. V of [34], which we recommend to get details about the physical meaning of the action integral and its extremals). In view of these phenomena, and because the

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2010 *Mathematics Subject Classification.* 49K15, 49S05, 34K11.

*Key words and phrases.* Stationary action; Functional extrema; conjugate points; oscillatory solutions; Lane-Emden equations.

emphasis was on the equations of motion, the elucidation of the true nature of the critical paths of the action functional was omitted, and the interest focused on the stationary property. However, some recent papers have made a “call to action” (the pun is not ours, see [66], [51], [34], [33]), renewing the interest in their extremal properties, not only their character of paths rendering the action stationary.

Our aim in this paper is twofold: on the one hand, to offer a concise, yet rigorous and self-contained, overview of some elementary techniques of non-linear analysis to investigate the extremals of an action functional. On the other hand, we intend to show several non trivial examples of physical interest illustrating the use of these techniques. We have avoided the well-known cases, so our examples go beyond oscillators and central potentials, and are taken from modern theories, ranging from astrophysics (Lane-Emden equations) to relativistic particles with energy dissipation. Each one of these examples has been chosen to illustrate some particular feature. Thus, example 6.1 shows a Lagrangian for a dissipative system; in example 6.2.3, as a bonus of the theory developed, we explicitly compute the solution (and its zeros) of an equation of the form  $y'' + q(x)^2 y = 0$  with  $q(x)$  rational; example 6.6 contains a justification of the use of Lagrange multipliers in the maximum entropy principle, etc.

We also differ from previous works, such as [34] or [33], in the flavour of the treatment: we feel that discussions trying to explain some plain analytic effects in physical terms are too lengthy, and sometimes add confusion instead of enlightenment when it comes to explicit computations. Thus, we center our exposition around the analytic definition and properties of the Gâteaux derivatives of functionals defined by integration (Lagrange functionals) and the techniques for the study of the behaviour of solutions of differential equations such as convexity or the comparison theorems of the Sturmian theory. This will be particularly patent in section 5, where we show that the main result in [34] is a direct consequence of well-known properties of the zeros of the Jacobi equation (see Proposition 5.1 and comments).

We offer short proofs for those results that seem to be not common in the physics literature. The bibliography, although by no means complete, is somewhat lengthy as a result of our efforts to make it useful.

## 2. CALCULUS OF VARIATIONS

In this section we will briefly describe some basic concepts in the calculus of variations in order to set up our notation and conventions, and also in order to introduce the Jacobi equation and conjugate points as explicit criteria for a given extremal solution to be a minimum. We will start by discussing Gâteaux derivatives and extrema of functionals. For general references on the topics of functional analysis and calculus of variations, see [16], [19], [23], [27], [60], [62], [69], [70], [76]. Note that we deal with the local aspects of the theory, exclusively. There are several approaches to the global setting, some of these were developed in [29], [24], [59]; more modern versions are developed in [45]. For detailed accounts of the theory involved in the global analysis, see [44] and [26].

**2.1. Gâteaux derivatives.** Let  $(E, \| \cdot \|)$  be a Banach space,  $D \subseteq E$  an open subset of  $E$  and  $y_0 \in D$ . Given a functional  $J : D \rightarrow \mathbb{R}$ , if  $v \in E$  is a non zero vector and  $|t|$  small enough,  $y_0 + tv$  will lie in  $D$  so the following definition makes sense.

**Definition 2.1.** *Whenever it exists, the limit*

$$\delta J(y_0, v) := \lim_{t \rightarrow 0} \frac{J(y_0 + tv) - J(y_0)}{t}$$

*is called the Gâteaux derivative (or first variation) of  $J$  at  $y_0$  in the direction  $v \in E$ . This defines a mapping  $\delta J(y_0, \cdot) : E \rightarrow \mathbb{R}$ . If this mapping is linear and continuous, we denote it by  $J'(y_0)$  and say that  $J$  is Gâteaux differentiable at  $y_0$ . Thus, under these conditions,  $\delta J(y_0, v) = J'(y_0)(v)$ . Another common notation is  $\delta J(y_0, v) = \delta_{y_0} J(v)$ .*

*The  $y_0 \in D$  such that  $J'(y_0) = 0$  are called critical points of the functional  $J$ .*

The extension to higher-order derivatives is immediate. If, for a fixed  $v \in E$ ,  $\delta J(z, v)$  exists for every  $z \in D$ , we have a mapping  $D \rightarrow \mathbb{R}$  and we can compute its Gâteaux derivative. Given an  $y_0 \in D$  and  $z, v \in E$ , the second Gâteaux derivative (or second variation) of  $J$  at  $y_0$  in the directions  $v$  and  $z$  (in that order) is

$$\delta^2 J(y_0, v, z) := \lim_{t \rightarrow 0} \frac{\delta J_{y_0+tz}(v) - \delta_{y_0} J(v)}{t}.$$

If  $\delta^2 J(y_0, v, z)$  exists for any  $z, v \in E$ , and  $(v, z) \mapsto \delta^2 J(y_0, v, z)$  is bilinear and continuous, we say that  $J$  is twice Gâteaux differentiable at  $y_0 \in D$ , and write  $J''(y_0)$  for this mapping. With these notations, we will write

$$\delta_{y_0}^2 J(v) = \delta^2 J(y_0, v) := \delta^2 J(y_0, v, v).$$

**Remark 2.2.** *For fixed  $y_0 \in D$  and  $v \in E$ , if we consider the function  $j_{(y_0, v)} : \mathbb{R} \rightarrow \mathbb{R}$  by  $j_{(y_0, v)}(t) = J(y_0 + tv)$ , it is obvious that it is defined in some open neighborhood of 0,  $] - \varepsilon, \varepsilon[$ , and the higher-order variations of  $J$  are given by*

$$\delta^n J(y_0, v) = j_{(y_0, v)}^{(n)}(0).$$

We will be interested in a particular class of functionals. To introduce it, we first need a technical observation: given an open subset  $U \subset \mathbb{R}^3$ , the set

$$D_U = \{y \in \mathcal{C}^1([a, b]) : \forall x \in [a, b], (x, y(x), y'(x)) \in U\}$$

(the prime denotes derivation, although we will also make free use of the physicist's dot notation for derivatives) is evidently contained in the Banach space  $(\mathcal{C}^1([a, b]), \|\cdot\|)$ , endowed with the norm

$$\|y\| = \|y\|_0 + \|y'\|_0,$$

where  $\|\cdot\|_0$  is the supremum norm. Moreover,  $D_U \subset \mathcal{C}^1([a, b])$  is an open subset. This follows from the fact that for a given  $y_0 \in D_U$  the set  $\{x, y_0(x), y'_0(x) : x \in [a, b]\}$  is compact, so it has an open neighborhood contained in  $U$ .

**Definition 2.3.** *A function  $L \in \mathcal{C}^2(U)$  is called a Lagrangian. To every Lagrangian it corresponds a functional  $J : D_U \rightarrow \mathbb{R}$ , called its action, defined by*

$$J(y) = \int_a^b L(x, y(x), y'(x)) dx.$$

**Proposition 2.4.** *For any  $U \subset \mathbb{R}^3$ , the action  $J$  is Gâteaux differentiable on  $D_U$ .*

*Proof.* Let  $y \in D_U$ . Taking into account the remark 2.2, note that for any  $t \in ] - \varepsilon, \varepsilon[$  (applying Leibniz's theorem of derivation under the integral):

$$j'_{(y, v)}(t) = \int_a^b \frac{d}{dt} (L(s, y(s) + tv(s), y'(s) + tv'(s))) ds.$$

Evaluating the derivative at  $t = 0$ , we get

$$\delta J(y, v) = \int_a^b (v(s)D_2L(s, y(s), y'(s)) + v'(s)D_3L(s, y(s), y'(s))) ds. \quad (1)$$

Note that  $\delta J(y, v)$  is linear in  $v$ . Moreover,

$$|\delta J(y, v)| \leq \|v\| \int_a^b (|D_2L(s, y(s), y'(s))| + |D_3L(s, y(s), y'(s))|) ds,$$

where the integral exists (because the  $D_iL$ ,  $i \in \{1, 2\}$ , are continuous on  $U$  and  $y, y'$  on  $[a, b]$ ) and it is a number depending only on  $y$ , thus constant for fixed  $y$ . Then,  $\delta J(y, v)$  is also continuous in  $v$ .  $\square$

## 2.2. Local extrema of functionals.

**Definition 2.5.** Let  $J : D \rightarrow \mathbb{R}$  be a functional and let  $y_0 \in D$ . We will say that  $J$  has a local maximum (local minimum, respectively) in  $y_0$  if for all  $y \in G$ , where  $G \subset D$  is a convex neighborhood of the point  $y_0$ , it follows that

$$J(y_0) \geq J(y)$$

( $J(y_0) \leq J(y)$ , respectively).

**Theorem 2.6.** Let  $J : D \rightarrow \mathbb{R}$  be a functional and  $y_0 \in D$ . Then:

- (1) (Necessary condition) If  $J$  has a local extremal and the variation  $\delta J(y_0, v)$  exists for some  $v \in E$ , then  $\delta J(y_0, v) = 0$ . Thus, if  $J$  is Gâteaux differentiable at  $y_0$ ,  $J'(y_0) = 0$ .
- (2) (Sufficient condition for a minimum) The functional  $J$  has a local minimum at  $y_0$  whenever the following hold:
  - (a) For each  $v \in E$ ,  $\delta J(y_0, v) = 0$ .
  - (b) (Coercivity) For any  $y$  in a convex neighborhood of  $y_0$ , the second variation  $\delta^2 J(y, v)$  exists for each  $v \in E$ . Moreover, there exists a  $c > 0$  such that

$$\delta^2 J(y_0, v) \geq c\|v\|^2,$$

for all  $v \in E$ .

- (c) (Weak continuity) Given  $\varepsilon > 0$ , there exists an  $\eta > 0$  such that

$$|\delta^2 J(y, v) - \delta^2 J(y_0, v)| \leq \varepsilon\|v\|^2,$$

for any  $v \in E$  and  $y$  satisfying  $\|y - y_0\| < \eta$ .

*Proof.*

- (1) Consider  $j_{(y_0, v)}(t) = J(y_0 + tv)$ , so if  $J$  has a local extremal at  $y_0$ ,  $j_{(y_0, v)}$  has a local extremum at  $t = 0$ . Then, it must be (recall remark 2.2)  $0 = j'_{(y_0, v)}(0) = \delta J(y_0, v)$ .
- (2) Suppose each of (2a), (2b), (2c) holds. As before, we have  $j'_{(y_0, v)}(t) = \delta J(y_0 + tv, v)$  and  $j''_{(y_0, v)}(t) = \delta^2 J(y_0 + tv, v)$ . The hypothesis on the second derivatives of  $J$  allows us to develop  $j_{(y_0, v)}(t)$  by Taylor in the interval  $[0, 1]$ , and there exists a  $\xi \in ]0, 1[$  such that

$$J(y_0 + v) - J(y_0) = j_{(y_0, v)}(1) - j_{(y_0, v)}(0) = \frac{1}{2}j''_{(y_0, v)}(\xi).$$

As  $j''_{(y_0, v)}(\xi) = \delta^2 J(y_0 + \xi v, v)$ , we have the following bound:

$$\begin{aligned} J(y_0 + v) - J(y_0) &= \\ \frac{1}{2} \delta^2 J(y_0, v) + \frac{1}{2} \left( \delta^2 J(y_0 + \xi v, v) - \frac{1}{2} \delta^2 J(y_0, v) \right) &\geq \\ \frac{1}{2} c \|v\|^2 + \frac{1}{2} \left( \delta^2 J(y_0 + \xi v, v) - \frac{1}{2} \delta^2 J(y_0, v) \right) \end{aligned}$$

Taking  $\varepsilon = c/4$  in (2c), there exists an  $\eta > 0$  such that

$$|\delta^2 J(y, v) - \delta^2 J(y_0, v)| \leq \frac{c}{4} \|v\|^2,$$

for  $\|y - y_0\| < \eta$ . Choosing now  $\|v\| < c\eta/2$ , it is  $\|y_0 + \xi v - y_0\| \leq \|v\| < c\eta/2$ , so

$$|\delta^2 J(y_0 + \xi v, v) - \delta^2 J(y_0, v)| \leq \frac{c}{4} \|v\|^2.$$

Substituting:

$$J(y_0 + v) - J(y_0) \geq \frac{1}{2} c \|v\|^2 - \frac{1}{4} c \|v\|^2 = \frac{1}{4} c \|v\|^2 > 0,$$

so  $J$  has a local minimum. □

**Remark 2.7.** Writing  $c < 0$  and reversing the inequality for  $\delta^2 J(y_0, v)$  in (2b), we get sufficient conditions for a local maximum.

**Remark 2.8.** Note that condition (2a) alone does not guarantee the existence of a local minimum, see counter-examples in [53], §2.10.

It is interesting to particularize the condition  $\delta J(y_0, v) = 0$  to the case of an action functional. For this, we need a couple of technical results whose proof is straightforward, but anyway can be found in any of the references cited at the beginning of this section. We will denote  $\mathcal{C}_0^k([a, b]) = \{f \in \mathcal{C}^k([a, b]) : f(a) = 0 = f(b)\}$ .

**Lemma 2.9** (Lagrange). *Let  $f \in \mathcal{C}([a, b])$  be a continuous real-valued function over the interval  $[a, b]$  such that*

$$\int_a^b f(x) \mu(x) \, dx = 0$$

*for all  $\mu \in \mathcal{C}_0([a, b])$ . Then it follows  $f \equiv 0$ .*

**Lemma 2.10** (DuBois-Reymond). *Let  $f \in \mathcal{C}([a, b])$  and  $g \in \mathcal{C}^1([a, b])$  such that*

$$\int_a^b (f(x) \mu(x) + g(x) \mu'(x)) \, dx = 0$$

*for all  $\mu \in \mathcal{C}_0^1([a, b])$ . Then it follows  $g' = -f$ .*

Now, a simple integration by parts in (1), and the application of Lemmas 2.9, 2.10 and Theorem 2.6, leads directly to the following result.

**Theorem 2.11** (Euler-Lagrange). *If  $y \in D$  is an extremal (maximum or minimum) for the action functional  $J : D \rightarrow \mathbb{R}$  given as in Definition 2.3, then  $y$  must satisfy the Euler-Lagrange equations*

$$D_2L(x, y(x), y'(x)) - \frac{d}{dx}D_3L(x, y(x), y'(x)) = 0. \quad (2)$$

**Remark 2.12.** *In physics literature, it is common to commit a slight abuse of notation and to write the Euler-Lagrange equations in the form*

$$\frac{\partial}{\partial y}L(x, y(x), y'(x)) - \frac{d}{dx} \frac{\partial}{\partial y'}L(x, y(x), y'(x)) = 0.$$

Note that, for the case at hand, writing  $L(s, y(s) + tv(s), y'(s) + tv'(s)) = L(s)$  for simplicity:

$$j''_{(y,v)}(t) = \int_a^b (v^2(s)D_{22}L(s) + 2v(s)v'(s)D_{23}L(s) + (v')^2(s)D_{33}L(s))ds,$$

so, evaluating at  $t = 0$ ,

$$\begin{aligned} \delta^2 J(y, v) &= \int_a^b (v^2(s)D_{22}L(s, y(s), y'(s)) \\ &+ 2v(s)v'(s)D_{23}L(s, y(s), y'(s)) + (v')^2(s)D_{33}L(s, y(s), y'(s)))ds. \end{aligned} \quad (3)$$

It is now a routine computation (continuity arguments and Schwarz inequality) to prove that for an action functional  $J(y) = \int_a^b L(x, y, y')dx$  with  $L \in \mathcal{C}^2(U)$  such that its second partial derivatives are bounded on  $U$ , under the hypothesis (2a) and (2b) of Theorem 2.6, the condition (2c) is satisfied ([27], pg. 224). Thus, a path  $y_0 \in D$  is a minimum if it satisfies Euler-Lagrange's equations (2) and the second Gâteaux differential at  $y_0$  is coercive, that is, there exists a  $c > 0$  such that, for all  $v \in E$ :

$$\delta^2 J(y_0, v) \geq c\|v\|^2. \quad (4)$$

**2.3. Problems with constraints.** The calculus of variation is frequently applied when there are constraints. The problem can be reduced to that of extremizing a single functional constructed out from the original one and the constraints.

**Definition 2.13.** *Let  $J$  be a functional defined on a Banach space  $(X, \|\cdot\|)$ . We say that  $\delta J$  is weakly continuous at  $y \in X$  if:*

- (a) *The domain of  $J$  contains an open neighborhood  $D \ni y$ , and, for each  $h \in X$ , the variation  $\delta J(y, h)$  is defined.*
- (b)  $\lim_{z \rightarrow y} \delta J(z, h) = \delta J(y, h)$ .

*If there exists an  $r > 0$  such that  $\delta J$  is weakly continuous for every  $z \in B(y; r)$ , we say that  $\delta J$  is locally weakly continuous at  $y$ , or simply weakly continuous near  $y$ .*

When we have an open subset  $U = ]a, b[ \times \mathbb{R} \times \mathbb{R} \subset \mathbb{R}^3$  and an action functional  $J : D_U \rightarrow \mathbb{R}$ ,  $J(y) = \int_a^b L(x, y, y')dx$ , it is easy to see that imposing some mild conditions on the Lagrangian  $L \in \mathcal{C}^2(U)$  we obtain a weakly continuous functional. For instance, it is enough to require that the second partial derivatives of  $L$  be bounded on  $U$ , or that its first partial derivatives be uniformly continuous. For most of the actions appearing in physics, however, it is usually easier to prove the weak continuity directly from the definition (cfr. Example 6.6).

Let now  $J =: K_0, K_1, \dots, K_r$  be functionals defined on  $D$ , all of them Gâteaux differentiable at each point  $y \in D$ . We will assume that the set

$$S = \{y \in D : K_i(y) = k_i, 1 \leq i \leq r\}$$

is not void, and that  $y_0$  is an interior point of  $S$  such that  $J|_S$  has a local extremal at  $y_0$ .

**Proposition 2.14.** *Let  $\delta K_j$  be weakly continuous near  $y_0$ ,  $0 \leq j \leq r$ . Then, for any  $h =: h_0, h_1, \dots, h_r \in X$  we have:*

$$\det(\delta K_j(y_0, h_m)) = 0, \quad 0 \leq j, m \leq r.$$

*Proof.* By reduction to the absurd. Let us assume that there exist  $h, h_1, \dots, h_r$  such that the determinant is non zero. As  $y_0$  has an open neighborhood  $D$ , there exist a set of scalars  $\alpha, \beta_1, \dots, \beta_r$  such that  $y_0 + \alpha h + \beta_1 h_1 + \dots + \beta_r h_r \in D$ , and the variations  $\delta J, \delta K_1, \dots, \delta K_r$  are continuous at  $y_0 + \alpha h + \beta_1 h_1 + \dots + \beta_r h_r$ . Let us define the function  $F : \mathbb{R}^{r+1} \rightarrow \mathbb{R}^{r+1}$ , on a neighborhood  $G$  of the origin  $(0, \dots, 0) \in \mathbb{R}^{r+1}$ , by

$$F_{p+1}(\alpha, \beta_1, \dots, \beta_r) = K_p(y_0 + \alpha h + \beta_1 h_1 + \dots + \beta_r h_r),$$

for  $0 \leq p \leq r$  (remember  $K_0 = J$  and  $h_0 = h$ ). It is immediate that

$$D_{q+1}F_{p+1}(\alpha, \beta_1, \dots, \beta_r) = \delta K_p(y_0 + \alpha h + \beta_1 h_1 + \dots + \beta_r h_r, h_q),$$

for  $0 \leq q \leq r$ . Now, as  $\delta J, \delta K_1, \dots, \delta K_r$  are continuous near  $y_0$ , by shrinking  $G$  if necessary we can assume  $F \in \mathcal{C}^1(G)$ , with Jacobian at the origin:

$$\det(\delta K_j(y_0, h_m)) \neq 0, \quad 0 \leq j, m \leq r$$

by hypothesis. Applying to  $F$  the inverse function theorem in the neighborhood of the origin, we get that there exists an open subset  $V \subset \mathbb{R}^{r+1}$ , containing  $F(0, \dots, 0) = (J(y_0), k_1, \dots, k_r)$ , and a local diffeomorphism  $\varphi : V \rightarrow G$  such that  $\tilde{G} = \varphi(V) \subset G$  is an open neighborhood of the origin in  $\mathbb{R}^{r+1}$ , and for all  $(x, y_1, \dots, y_r) \in V$ :

$$(x, y_1, \dots, y_r) = F(\varphi(x, y_1, \dots, y_r)).$$

In particular,  $\varphi(J(y_0), k_1, \dots, k_r) = (0, 0, \dots, 0)$ . As  $(J(y_0), k_1, \dots, k_r)$  is a point of the open set  $V \subset \mathbb{R}^{r+1}$ , we can find in  $V$  two different points  $(x_1, k_1, \dots, k_r)$  and  $(x_2, k_1, \dots, k_r)$  such that  $x_1 < J(y_0) < x_2$ . Their corresponding images by  $\varphi$  are  $(\alpha^1, \beta_1^1, \dots, \beta_r^1) = \varphi(x_1, k_1, \dots, k_r)$  and  $(\alpha^2, \beta_1^2, \dots, \beta_r^2) = \varphi(x_2, k_1, \dots, k_r)$ . Thus, the vectors  $u = y_0 + \alpha^1 h + \beta_1^1 h_1 + \dots + \beta_r^1 h_r$  and  $v = y_0 + \alpha^2 h + \beta_1^2 h_1 + \dots + \beta_r^2 h_r$  belong to  $D$ . Moreover,

$$F(\alpha^1, \beta_1^1, \dots, \beta_r^1) = (x_1, k_1, \dots, k_r),$$

$$F(\alpha^2, \beta_1^2, \dots, \beta_r^2) = (x_2, k_1, \dots, k_r),$$

so, equating components:

$$J(u) = K_0(u) = F_0(\alpha^1, \beta_1^1, \dots, \beta_r^1) = x_1,$$

$$K_p(u) = F_p(u) = F_p(\alpha^1, \beta_1^1, \dots, \beta_r^1) = k_p,$$

for  $1 \leq p \leq r$ , and

$$J(v) = K_0(v) = F_0(\alpha^2, \beta_1^2, \dots, \beta_r^2) = x_2,$$

$$K_p(v) = F_p(v) = F_p(\alpha^2, \beta_1^2, \dots, \beta_r^2) = k_p,$$

so  $u, v \in S$  too. But, because of our choices,  $J(u) = x_1 < J(y_0) < x_2 = J(v)$ , and this construction can be repeated for  $x_1, x_2$  arbitrarily close to  $J(y_0)$ , so (by the continuity of  $\varphi$ ) the corresponding vectors  $u, v$  will be arbitrarily close to  $y_0$ , contradicting the assumption that  $y_0$  is a local extremal of  $J|_S$ .  $\square$

**Theorem 2.15.** *Let  $\delta K_j$  be weakly continuous near  $y_0$ . Then, either*

$$\det(\delta K_i(y_0, h_l)) = 0, \quad 1 \leq i, l \leq r,$$

*or there exist a set of real numbers (the Lagrange multipliers)  $\lambda_1, \dots, \lambda_r$  such that*

$$\delta J(y_0, h) = \sum_{i=1}^m \lambda_i \delta K_i(y_0, h).$$

*Proof.* By Proposition 2.14,  $\det(\delta K_j(y_0, h_j)) = 0$ ,  $0 \leq j \leq r$ ; developing by the first column (where  $K_0 = J$  and  $h_0 = h$ ), we get

$$\delta J(y_0, h) \det(\delta K_i(y_0, h_l)) + \sum_{i=1}^m \mu_i \delta K_i(y_0, h) = 0, \quad (5)$$

for some set of scalars  $\mu_1, \dots, \mu_r$  (which depend on the vectors  $(h_1, \dots, h_r)$ ). Then, either

$$\det(\delta K_i(y_0, h_l)) = 0, \quad 1 \leq i, l \leq r,$$

or there exist some set of vectors  $v_1, \dots, v_r \in X$  such that  $\det(\delta K_i(y_0, v_l)) \neq 0$ . In this case, substituting in (5), we obtain

$$\delta J(y_0, h) = \sum_{i=1}^m \lambda_i \delta K_i(y_0, h),$$

where

$$\lambda_i = -\frac{\mu_i}{\det(\delta K_i(y_0, h_l))}.$$

$\square$

### 3. CONJUGATE POINTS

Let us rewrite the conditions for a minimum of the action  $J$ , found in the previous section, in terms of a differential equation involving the derivatives of the Lagrangian  $L$ .

**Proposition 3.1.** *Let the action  $J$  be given as in Definition 2.3, and let  $y_0 \in D$ . Then, the second variation of  $J$  at  $y_0$ , in the direction of a  $v \in \mathcal{C}_0^1([a, b])$ ,  $\delta^2 J(y_0, v)$ , reads*

$$\delta^2 J(y_0, v) = \frac{1}{2} \int_a^b (Pv'^2 + Qv^2) \, dx, \quad (6)$$

*where the functions  $P(x)$  and  $Q(x)$  are explicitly given by*

$$P(x) = D_{33}L(x, y(x), y'(x)) \quad (7)$$

$$Q(x) = D_{22}L(x, y(x), y'(x)) - \frac{d}{dx} D_{23}L(x, y(x), y'(x)) \quad (8)$$

*Proof.* Just make an integration by parts in the middle term of the integrand in (3), taking into account the boundary conditions on  $v \in \mathcal{C}_0^1([a, b])$ .  $\square$



**Remark 3.2.** *The notation used in physics is:*

$$P = \frac{\partial^2 L}{\partial (y')^2},$$

$$Q = \frac{\partial^2 L}{\partial y^2} - \frac{d}{dx} \left( \frac{\partial^2 L}{\partial y \partial y'} \right).$$

Lagrange considered equation (6) already in 1786. He thought that a sufficient condition to have a minimum would be the positivity of the second variation  $\delta^2 J(y_0, v) > 0$  (which is *not* true: coercivity is needed), so he tried to “complete the square” in (6) by introducing a boundary term of the form  $gv^2/2$ , where  $g \in \mathcal{C}([a, b])$  is to be determined. In this way, we have

$$\begin{aligned} \delta^2 J(y_0, v) &= \int_a^b (P(v')^2 + Qv^2) \, dx + \int_a^b \frac{d}{dx}(gv^2) \, dx \\ &= \int_a^b (P(v')^2 + 2gvv' + (g' + Q)v^2) \, dx \\ &= \int_a^b P \left( v' + \frac{g}{P}v \right)^2 \, dx + \int_a^b \left( g' + Q - \frac{g^2}{P} \right) v^2 \, dx. \end{aligned} \tag{9}$$

Thus, it is straightforward that  $\delta^2 J(y_0, v)$  will be positive definite if the following conditions are satisfied

$$P = D_{33}L(x, y(x), y'(x)) > 0, \tag{10}$$

$$g' + Q - \frac{g^2}{P} = 0 \text{ has a solution } g. \tag{11}$$

Condition (10) is known as the Legendre condition. Also, note that equation (11) is of Riccati type. This equation is basic to determine the extremality properties of critical points of the action.

**Definition 3.3.** *Let  $J$  be an action functional, and let  $f \in \mathcal{C}([a, b])$ . The differential equation for  $f$*

$$-\frac{d}{dx} \left( P \frac{df}{dx} \right) + Qf = 0, \tag{12}$$

*where  $P$  and  $Q$  are given in (7) and (8), respectively, is called the Jacobi equation.*

Notice that Jacobi equation is simply obtained by introducing the change of variable  $g = -Pd(\ln f)/dx$  in equation (11), which renders it linear. Once we solve the equation for  $f$ , we get  $g$  and then we can assure that, if  $P > 0$ , then  $\delta^2 J(y_0, v) > 0$ . Although we know that this is not enough to guarantee a minimum (recall Remark 2.8), the properties of the solutions to the Jacobi equation (12) will lead us to an equivalent condition for a minimum of the action, given in terms of quantities determined by the Lagrangian.

**Definition 3.4.** *Two points  $p, q \in \mathbb{R}$  (with  $p < q$ ) are called conjugate with respect to the Jacobi equation (12) if there is a solution  $f \in \mathcal{C}^2([a, b])$  of (12) such that  $f|_{[p, q]} \neq 0$  and  $f(p) = 0 = f(q)$ .*

The following result is just a particular case of K. Friedrichs’ inequalities for the one-dimensional case (see [1]). Its proof can also be done directly, as an application of the Schwarz inequality.

**Lemma 3.5.** *For any  $v \in \mathcal{C}_0^1([a, b])$ , we have*

$$\int_a^b v^2(x) dx \leq \frac{(b-a)^2}{2} \int_a^b (v')^2(x) dx$$

**Theorem 3.6.** *Let  $J(y)$  be an action functional as in Definition 2.3. Sufficient conditions for a critical point  $y_0$  of  $J(y)$  to be a local minimum in the interval  $[a, b]$  are given by*

(a) *For all  $x \in [a, b]$*

$$P(x) = D_{33}L(x, y(x), y'(x)) > 0.$$

(b) *The interval  $[a, b]$  does not contain conjugate points at  $x = a$  with respect to Jacobi equation (12).*

*Proof.* Recall from (10) the expression

$$\delta^2 J(y_0, v) = \int_a^b (P(v')^2 + 2g v v' + (g' + Q)v^2) dx.$$

Because of the assumptions made on the continuity of the derivatives of  $L$  and the compactness of  $[a, b]$ , we can choose a number  $\sigma$  such that  $0 \leq \sigma < \min_{[a, b]} \{P(x)\}$ . Inserting  $\sigma P(v')^2 - \sigma P(v')^2$  in the equation above, and repeating the computation in (10) gives

$$\begin{aligned} \delta^2 J(y_0, v) &= \int_a^b (P - \sigma) \left( v' + \frac{g}{P - \sigma} \right)^2 dx \\ &+ \int_a^b \left( g' + Q - \frac{g^2}{P - \sigma} \right) v^2 dx + \sigma \int_a^b (v')^2 dx. \end{aligned} \quad (13)$$

As  $P(x) > 0$  and we have chosen  $\sigma$  such that  $P(x) - \sigma > 0$  on  $[a, b]$ , the first integral is positive, as it is the third one. In order to cancel out the second integral, we must take a  $g \in \mathcal{C}^1([a, b])$  such that

$$g' + Q - \frac{g^2}{P - \sigma} = 0.$$

Introducing a function  $f \in \mathcal{C}^2([a, b])$  through

$$g = -\frac{f'}{f}(P - \sigma), \quad (14)$$

we arrive at the equation for  $f$ :

$$-\frac{d}{dx} \left( (P - \sigma) \frac{df}{dx} \right) + Qf = 0. \quad (15)$$

By the theorem on the dependence on parameters of the solutions to a second order differential equation, the general solution of (15) can be written as  $f(x, \sigma)$ , with  $f(x, 0) = f(x)$ . Note that, by hypothesis,  $f(x, 0)$  does not admit points conjugate to  $a$  in  $[a, b]$ , so (by continuity), neither does  $f(x, \sigma)$  for  $\sigma > 0$  but close enough to 0. If  $\tilde{f}(x) = f(x, \sigma)$  is such a solution, by substituting the corresponding  $\tilde{g}$  of

equation (14) into (13), we get:

$$\begin{aligned}\delta^2 J(y_0, v) &= \int_a^b (P - \sigma) \left( v' + \frac{\tilde{g}}{P - \sigma} \right)^2 dx + \sigma \int_a^b (v')^2 dx \\ &\geq \sigma \int_a^b (v')^2 dx.\end{aligned}$$

Now, applying Lemma 3.5,

$$\delta^2 J(y_0, v) \geq \frac{\sigma}{1 + \frac{(b-a)^2}{2}} \|v\|^2 := c \|v\|^2.$$

The statement follows then from Theorem 2.6 (see also the comments at the end of subsection 2.2).  $\square$

**Remark 3.7.** *We can obtain a criterion for a local maximum just by considering the condition  $P(x) = D_{33}L(x, y(x), y'(x)) < 0$  and repeating the computations in the theorem with the inequalities reversed.*

**Remark 3.8.** *In order to apply the criterion of conjugate points in practice, it is desirable to have at our disposal some tools for explicitly computing solutions of the Jacobi equation (12). An old (but useful) method ([53] pp. 56–57, [21] pp. 42–43) is the following: the general solution of Euler-Lagrange’s equations (which are second order) has the form  $y = y(x; \alpha, \beta)$ , where  $\alpha, \beta$  are constants of integration (on which the  $y$  dependence is differentiable, under some mild conditions. In the examples this will be obvious). Then,  $D_2y(x; \alpha, \beta) \equiv \frac{\partial y}{\partial \alpha}$  and  $D_3y(x; \alpha, \beta) \equiv \frac{\partial y}{\partial \beta}$  are two independent solutions of the Jacobi equation<sup>1</sup>. We will use this method in example 6.2.3.*

#### 4. CONVEX FUNCTIONALS

In this section we will discuss the particular case of convex Lagrangians. We will start by recalling that a subset  $S \subset \mathbb{R}^n$  is said to be convex if for all  $p, q \in S$  the interval  $[p, q]$  lies entirely inside of  $S$ . This is equivalent to say that

$$[p, q] := \{p + t(q - p) = tq + (1 - t)p : 0 \leq t \leq 1\} \subset S. \quad (16)$$

**Definition 4.1.** *A function  $f : S \subset \mathbb{R}^2$  defined over a convex set, is said to be convex if*

$$f([p, q]) \leq [f(p), f(q)]$$

*or, equivalently, for all  $t \in [0, 1]$ :*

$$f(p + t(q - p)) \leq f(p) + t(f(q) - f(p)) \quad (17)$$

Let  $f : S \rightarrow \mathbb{R}$  be a convex differentiable function. For any  $t \in [0, 1]$ ,  $p, q \in S$  we have (17), and, on the other hand, by the intermediate value theorem, there exists a  $w_t \in [p, p + t(q - p)]$  such that

$$f(p + t(q - p)) = f(p) + t d_{w_t} f(q - p), \quad (18)$$

---

<sup>1</sup>The proof is extremely simple: just take derivatives with respect to, say,  $\alpha$  in Euler-Lagrange’s equations (applying the chain rule) and collect terms, taking into account that

$$\frac{d}{dx} \left( \frac{\partial^2 L}{\partial y \partial y'} \frac{\partial y}{\partial \alpha} \right) = \frac{d}{dx} \left( \frac{\partial^2 L}{\partial y \partial y'} \right) \frac{\partial y}{\partial \alpha} + \frac{\partial^2 L}{\partial y \partial y'} \frac{\partial y'}{\partial \alpha}.$$

where  $d_{w_t}f$  denotes the differential of  $f$  at  $w_t$ . From (17) and (18) we get  $f(p) + td_{w_t}f(q-p) \leq tf(q) + (1-t)f(p)$ , that is:

$$f(p) + d_{w_t}f(q-p) \leq f(q).$$

Taking the limit  $t \rightarrow 0$  implies  $w_t \rightarrow p$ , so:

$$f(p) + d_p f(q-p) \leq f(q). \quad (19)$$

**Remark 4.2.** *There exist well-known criteria (in terms of the Hessian matrix) to decide whether a function  $f : S \rightarrow \mathbb{R}$  is convex or not, see for instance [70] sec. 10.7. We will apply one such criterion in Example 6.2.1.*

As a straightforward application of these results, we have the following theorem, stating that the critical points of an action with a convex Lagrangian are always minimals.

**Theorem 4.3.** *Consider a set  $U = [a, b] \times S \subset \mathbb{R}^3$ , such that for each fixed  $x \in [a, b]$  the set  $S_x = \{(x, u, v) \in U\} \subset \mathbb{R}^2$  is convex. Suppose that for any  $x \in [a, b]$ , the Lagrangian function  $L(x, \cdot, \cdot) : S_x \rightarrow \mathbb{R}$  is convex. Then, any critical path  $y_0 = y_0(x)$  is a minimal solution, among the paths with the same endpoints, for the corresponding action functional  $J(y) = \int_a^b L(x, y(x), y'(x))dx$ .*

*Proof.* The hypothesis of convexity implies, by (19),

$$\begin{aligned} L(x, u_2, v_2) &\geq L(x, u_1, v_1) + d_{(x, u_1, v_1)} L((u_2, v_2) - (u_1, v_1)) \\ &= L(x, u_1, v_1) + D_1 L(x, u_1, v_1)(u_2 - u_1) + D_2 L(x, u_1, v_1)(v_2 - v_1). \end{aligned} \quad (20)$$

Now we compute the action on a critical path,  $y_0(x)$ , and an arbitrary nearby one  $y(x)$ , with the same endpoints ( $y_0(a) = y(a)$  and  $y_0(b) = y(b)$ ), and compare them. From (20):

$$\begin{aligned} J(y) - J(y_0) &= \int_a^b (L(x, y(x), y'(x)) - L(x, y_0(x), y'_0(x)))dx \\ &\geq \int_a^b (D_1 L(x, y_0, y'_0)(y - y_0) + D_2 L(x, y_0, y'_0)(y' - y'_0))dx. \end{aligned} \quad (21)$$

The second term in the last integrand can be –as usual– integrated by parts, we then get:

$$J(y) - J(y_0) \geq \int_a^b \left( D_1 L(x, y_0, y'_0) - \frac{d}{dx} D_2 L(x, y_0, y'_0) \right) (y - y_0) dx.$$

But, by hypothesis,  $y_0(x)$  is a critical path; equivalently, for each  $x \in [a, b]$  it satisfies the Euler-Lagrange equations (2), and this implies  $J(y) \geq J(y_0)$ .  $\square$

**Remark 4.4.** *Reversing the inequalities, we obtain the corresponding result for concave functionals.*

## 5. STURMIAN THEORY

Sturmian theory is concerned with the analysis of the zeros that a solution of a linear second order differential equation, of the form

$$\frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = 0, \quad (22)$$

(with  $p(x), q(x)$  piece-wise continuous) has in a given interval of the independent variable. This theory is an invaluable tool to check the properties of critical points, as we will see in the examples of the next section.

The results stated here without proof are well-known (see [2], [61]). We write them just for easy reference.

As it is well-known, the differential equation (22) may be written, through the change of variable  $v = y \exp(\frac{1}{2} \int p dx)$ , in its normal form

$$\frac{d^2 v}{dx^2} + r(x)v = 0, \quad (23)$$

where  $r(x) = q(x) - \frac{1}{4}p^2(x) - \frac{1}{2}p'(x)$ , which clearly preserves the zeros of the solutions to (22) if  $\int^x p(s)ds$  is finite for finite  $x$ . The first observation is that the zeros of such an equation cannot accumulate.

**Proposition 5.1.** *Let  $y(x)$  be a non trivial solution of (22) or (23). Then, its zeros are simple and the set they form does not have accumulation points. Thus, on each closed interval  $[a, b]$ ,  $y(x)$  only possess a finite number of zeros.*

*Proof.* If  $x_0$  were a double zero,  $y(x_0) = 0 = y'(x_0)$ , so  $y(x)$  would be the trivial solution by uniqueness.

If  $x_0$  were an accumulation point for the zeros of  $y(x)$ , there would be a sequence of zeros  $(x_n)$  such that  $x_n \rightarrow x_0$ . By Rolle's theorem, there is a zero of the derivative between any two consecutive zeros of the function, so there would be a sequence  $(u_m)$  of zeros of  $y'(x)$  such that  $u_m \rightarrow x_0$ . Then, by continuity of both  $y$  and  $y'$ , we would have  $y(x_0) = 0 = y'(x_0)$ , which, as we have just seen, is impossible.  $\square$

As a corollary, the zeros of a non trivial solution of (22) or (23) are: either a finite set, or a sequence diverging to  $+\infty$ , or a sequence diverging to  $-\infty$ , or a sequence diverging to  $\pm\infty$ .

This applies in particular to the Jacobi equation (12). Thus, if we have a solution  $f(x)$  on the interval  $[a, b]$  such that  $f(a) = 0$ , its first zero after  $x = a$  must be located at a point  $c > a$ . In other words: there exists a  $c > a$  such that there are no conjugate points in the interval  $[a, c]$ . A direct consequence of this fact is that for a short enough interval, critical points of the action  $J(y) = \int_a^b L(x, y, y')dx$  such that  $P(x) = D_{33}L(x, y, y') > 0$ , are local minimizers.

Some authors state, erroneously, that *for any* Lagrangian the critical points are local minimizers. The origin of the confusion can be traced back to the fact that this is true for *natural* Lagrangians<sup>2</sup>, in particular it is true for free Lagrangians ( $V = 0$ ), for which the trajectories are geodesics. However, not every system of interest in Physics is natural. Of all the examples presented in this note, only Example 6.5 is natural; and in Example 6.6 we present a case for which the critical points are maximizers.

**Definition 5.2.** *If every solution  $y$  of (22) or (23) has arbitrarily large (in absolute value) zeros, then the equation (and all its solutions) are called oscillatory. Otherwise, the equation and all of its solutions are called non-oscillatory.*

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<sup>2</sup>That is, those of the form  $L(y, y') = K(y') - V(y)$  where  $K$  is a positive-definite quadratic form associated to some metric (usually  $K(y') = (y')^2/2$ , that is, the metric is the euclidean one) and  $V$  is some  $C^1$  function. Note that in this case  $P \geq 0$ .

**Theorem 5.3.** *Let  $y_1(x)$  and  $y_2(x)$  denote two linear independent solutions of equation (23). Then, the zeros of both functions are distinct and alternating in the sense that  $y_1(x)$  has a zero between any two consecutive zeros of  $y_2(x)$ , and vice-versa.*

**Theorem 5.4.** *If  $r(x) \leq 0$  on  $[a, b]$ , then no non-trivial solution of (23) can have two zeros on  $[a, b]$ .*

**Theorem 5.5** (Sturm's Comparison Theorem). *Let  $y_1(x)$  and  $y_2(x)$  be non-trivial solutions to the differential equation (23) with  $r_1(x)$  and  $r_2(x)$ , respectively. If  $r_1 > r_2$  in a certain interval  $[a, b]$ , then  $y_1(x)$  has at least a zero between two consecutive zeros of  $y_2(x)$ , unless  $y_1 = y_2$  on  $[a, b]$ .*

## 6. EXAMPLES

In this section we will develop some physically motivated examples in order to elucidate the ideas analysed so far. In each case, the regularity conditions on the Lagrangian are trivially satisfied. Unless otherwise explicitly stated, we will work on the space  $X = \mathcal{C}^1([a, b])$ . Also, in some examples we will follow the notation common in physics, taking  $t$  as the independent variable and  $x, \dot{x} = dx/dt$  as the dependent ones.

**6.1. Driven harmonic oscillator.** As it is well known, the differential equation for a driven damped harmonic oscillator under a sinusoidal external force is given by [30], [36], [41], [67], [68]:

$$\ddot{x} + \beta\dot{x} + \omega_0^2 x = \sin(\omega t). \quad (24)$$

Its solutions are well-known. They have the form

$$x(t) = \frac{1}{\omega Z} \sin(\omega t + \varphi), \quad (25)$$

where  $Z = \sqrt{\beta^2 + \frac{1}{\omega^2}(\omega_0^2 - \omega^2)}$  is the impedance and  $\varphi = \arctan\left(\frac{\beta\omega}{\omega_0^2 - \omega^2}\right)$  is the phase.

The corresponding Lagrangian function is:

$$L(t, x, \dot{x}) = \frac{1}{2}e^{\beta t} \left( \dot{x}^2 - \frac{\beta \sin(\omega t) - \omega \cos(\omega t)}{\omega^2 + \beta^2} \dot{x} - \omega_0^2 x^2 \right).$$

Here we follow the standard conventions and denote  $\omega_0 \in \mathbb{R}$  as the natural oscillation frequency,  $\beta \in \mathbb{R}^+$  is the damping parameter, and  $\omega \in \mathbb{R}$  stands for the frequency of the driving force. For this Lagrangian, we straightforwardly note that the functions  $P = e^{\beta t} > 0$  and  $Q = -\omega_0^2 e^{\beta t}$ , given by (7) and (8), respectively, lead to the Jacobi equation

$$\frac{d^2 f}{dt^2} + \beta \frac{df}{dt} + \omega_0^2 f = 0 \quad (26)$$

which is the damped harmonic oscillation equation for the function  $f$ . Note that this equation is damped by the same amount as the driven equation (24).

This is a second-order linear equation of constant coefficients, so it can be analytically solved and we get the general solution (for the underdamped<sup>3</sup> case  $\beta < \omega_0$ )

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<sup>3</sup>The other cases (critical damping and overdamping) are treated similarly.

in the form

$$f(t) = e^{-\frac{\beta t}{2}} \left( k_1 \sin \left( \frac{\sqrt{4\omega_0^2 - \beta^2} t}{2} \right) + k_2 \cos \left( \frac{\sqrt{4\omega_0^2 - \beta^2} t}{2} \right) \right),$$

where  $k_1, k_2$  are constants of integration. The solution verifying  $f(0) = 0$  can be written as

$$f(t) = C \frac{2}{\sqrt{4\omega_0^2 - \beta^2}} e^{-\frac{\beta t}{2}} \sin \left( \frac{\sqrt{4\omega_0^2 - \beta^2} t}{2} \right),$$

from which we clearly see that its zeros are located at the values  $t = \frac{2k\pi}{\sqrt{4\omega_0^2 - \beta^2}}$ , for  $k \in \mathbb{Z}$ . The first zero after  $t = 0$  is located at  $t = \frac{2\pi}{\sqrt{4\omega_0^2 - \beta^2}}$ , so the solution (25) is a minimum on the interval  $[0, \frac{2\pi}{\sqrt{4\omega_0^2 - \beta^2}}[$ . When  $\beta = 0$  this interval becomes  $[0, \frac{\pi}{\omega_0}[$ , which is the particular case of the harmonic oscillator (see [34], pg. 446).

**6.2. Lane-Emden Equations.** The Lane-Emden second-order differential equation was originally proposed by Lane [46], and studied in detail by Emden [17] and Fowler [20], in order to understand equilibrium configurations of spherical clouds of gas (self-gravitating polytropic gas spheres) [7], [13], [15]. Lane-Emden equations also appears in several other contexts such as viscous fluid dynamics, radiation, condensed matter, relativistic mechanics, and even for systems under chemical reactions (see [74], [28], and references therein for an account of its applications. For a mathematical treatment of their zeros, see [14]). The Lane-Emden equation is characterized by a non-linear term  $y^n(x)$ , where the non-negative parameter  $n \in \mathbb{Z}$  (the polytropic index, in its original context) defines the nature of the second-order differential equation

$$\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{dy}{dx} \right) + y^n = 0, \quad (27)$$

which may be obtained from the associated Lagrangian

$$L(x, y, y') = x^2 \left( \frac{y'^2}{2} - \frac{y^{n+1}}{n+1} \right), \quad (28)$$

in the sense that its Euler-Lagrange equations reduce to (27). For this Lagrangian, the functions (7) and (8) are given by  $P = x^2$  and  $Q = -nx^2y^{n-1}$ , respectively. As in the preceding example, the function  $P$  is always positive definite on any interval of the form  $]0, b]$ ,  $b \in \mathbb{R}$ . We also note that the function  $Q$  depends on the parameter  $n$ . In this way, we obtain the Jacobi equation (12)

$$\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{df}{dx} \right) + ny^{n-1}f = 0. \quad (29)$$

As the most frequent analytical solutions to the Lane-Emden equation are those corresponding to the values  $n = 0, 1, 5$ , [74] (but see [28] for other cases), we will focus next on solution of both, Euler-Lagrange and Jacobi equations, for these cases.

6.2.1.  $n=0$ . Consider any interval  $[0, b]$ . For this case, the general solution  $y(x)$  of Euler-Lagrange's equation (27) reads

$$y(x) = -\frac{x^2}{6} + \frac{k_2}{x} + k_1,$$

where  $k_1, k_2$  are arbitrary integration constants. Note that this function is singular at the origin. The physical origin of the problem demands that the solution verify  $y(0) = 1, y'(0) = 0$ , so we must take  $k_2 = 0, k_1 = 1$ . The solution is then

$$y(x) = -\frac{x^2}{6} + 1.$$

In this very simple case, it is not necessary to deal with the Jacobi equation, as the corresponding Lagrangian

$$L(x, y, y') = x^2 \left( \frac{y'^2}{2} - y \right),$$

determines, for each  $x \in [a, b]$  fixed, a function  $L(x, \cdot, \cdot)$  which is convex on the convex set  $\mathbb{R} \times \mathbb{R}$ , as it has a semi-definite Hessian: for any  $u, v \in \mathbb{R}$ ,

$$\det \text{Hess} L(x, u, v) = 0$$

$$D_{22} L(x, u, v) = 0$$

$$D_{33} L(x, u, v) = x^2 \geq 0.$$

Thus, the solution is minimal on  $]0, b]$ , for any  $b > 0$ .

6.2.2.  $n=1$ . The general solution the Lane-Emden equation (27) is given by

$$y(x) = k_1 \frac{\sin(x)}{x} + k_2 \frac{\cos(x)}{x} \quad (30)$$

where  $k_1, k_2$  are constants. Again, the physical meaning of the problem imposes the condition that the solutions be defined at  $x = 0$  (where it must be  $y(0) = 1$ ), so we must take  $k_1 = 1, k_2 = 0$ , getting the sinc function

$$y(x) = \frac{\sin(x)}{x}. \quad (31)$$

As the functions  $P = x^2, Q = -x^2$  in this case, Jacobi equation (29) results again a Lane-Emden equation with  $n = 1$  for the function  $f(x)$ . Hence, for  $n = 1$  solutions of the Jacobi equation and the Lane-Emden equation are of an identical nature. Thus, in view of (30), the solutions of the Jacobi equation defined for  $x = 0$  are those of the form  $f(x) = C \text{sinc}(x)$ . They have zeros located at the points  $x = k\pi, k \in \mathbb{Z} - \{1\}$ . So, on any interval of the form  $](k-1)\pi, k\pi[, k \in \mathbb{Z} - \{0, 1\}$ , the solutions (31) are minimal. A minimum is also obtained on  $]-\pi, 0[$  and  $]0, \pi[$ .

6.2.3.  $n=5$ . Analogously, in this case, the general solution to (27) reads (see [13], [39]<sup>4</sup>):

$$y(x; \alpha, \beta) = \sqrt{\frac{\alpha}{\frac{(\alpha x)^2}{\beta} + \frac{\beta}{3}}}. \quad (32)$$

---

<sup>4</sup>Although in these references only a 1-parameter family of solutions is given, it is easy to trace back the missing parameter  $\alpha$  from the calculations presented there (it is fixed at certain point to make the output of an integral more manageable).



We further note that if we impose the boundary conditions  $y(0) = 1$  and  $y'(0) = 0$  (which set  $\alpha = 1$ ,  $\beta = 3$ ) our solution becomes the common one

$$y(x; 1, 3) = \frac{\sqrt{3}}{\sqrt{3+x^2}}. \quad (33)$$

We will work on this simplified solution. For this case, the function  $Q = -5x^2y^4$ , thus yielding the Jacobi equation

$$\frac{d^2f}{dx^2} + \frac{2}{x} \frac{df}{dx} + \frac{45}{(3+x^2)^2} f = 0, \quad (34)$$

which may be simplified to its normal form (23) by the change  $f(x) = u(x)/x$ :

$$\frac{d^2u}{dx^2} + \frac{45}{(3+x^2)^2} u = 0. \quad (35)$$

Let us apply the method outlined in Remark 3.8. The derivative of the general solution with respect to the parameter  $\alpha$ , evaluated at the values that give the solution we are considering, is

$$D_2L(x; 1, 3) = -\frac{\sqrt{3}(x^2-3)}{2(x^2+3)^{\frac{3}{2}}}.$$

It is immediate to check that this is indeed a solution of Jacobi's equation (34). The derivative with respect to  $\beta$  gives nothing new (a multiple of  $D_2L(x; 1, 3)$ ). By making the change of variables stated above, we get:

$$u(x) = -\frac{\sqrt{3}x(x^2-3)}{2(x^2+3)^{\frac{3}{2}}}.$$

Note that  $u(0) = 0$ . The first (and only) zero of  $u(x)$  after  $x = 0$  is given by  $x = \sqrt{3}$ . Thus, the solution to the Lane-Emden equation for  $n = 5$  (33), is a minimum for  $x \in [0, \sqrt{3}]$ .

**6.3. Quantum gravity in one dimension.** In modern physics, spin foams models have been introduced in order to analyse certain generalizations of path integrals appearing in gauge theories. In particular, in quantum gravity the spin foam approach has been developed as a tool to understand dynamical issues of the theory by the introduction of discretizations describing the metric properties of spacetime [3]. To some extent, spin foams for quantum gravity were motivated by a particular discretization of general relativity known as Regge calculus [73]. In this context, a discrete model for a scalar field representing gravity in one temporal dimension has been studied in detail in [35]. Here we present the continuum analogue of this model. The general action functional is:

$$J(y) = \frac{1}{2} \int \sqrt{g(x)} (g^{-1}(x)y'(x)^2 + \omega y^2(x)) dx,$$

where  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a positive function which acts as the metric on  $\mathbb{R}$ , and will be taken in what follows as  $g(x) = \exp(x)$ , for simplicity. Thus our model Lagrangian will be

$$L(x, y, y') = \frac{1}{2} \exp(x/2) (\exp(-x)(y')^2 + \omega y^2).$$

The Euler-Lagrange equations readily follow:

$$\omega \exp(x/2)y + \frac{1}{2} \exp(-x/2)y' - \exp(-x/2)y'' = 0,$$

or, as  $\exp(-x/2) > 0$ ,

$$y'' - \frac{1}{2}y' - \omega e^x y = 0. \quad (36)$$

By making the change of variable  $u = \exp(x/2)$ , we can put (36) in the form

$$u^2 \left( \frac{1}{4} \frac{dy}{du} - \omega y(u) \right) = 0,$$

which, as  $u = \exp(x/2) > 0$ , reduces to

$$\frac{dy}{du} - 4\omega y(u) = 0.$$

This equation is integrated by elementary techniques; its solutions are:

$$y(u) = k_1 \exp(2\sqrt{\omega}u) + k_2 \exp(-2\sqrt{\omega}u),$$

(with  $k_1, k_2$  constants of integration) or, in the original variable  $x$ :

$$y(x) = k_1 \exp(2\sqrt{\omega} \exp(x/2)) + k_2 \exp(-2\sqrt{\omega} \exp(x/2)). \quad (37)$$

The coefficients of the Jacobi equation are  $P = \exp(-x/2) > 0$  and  $Q = \omega \exp(x/2)$ , so the Jacobi equation is (after simplifying an  $\exp(x/2) > 0$  factor):

$$\frac{d^2 f}{dx^2} - \frac{1}{2} \frac{df}{dx} - \omega e^x f = 0,$$

which has the same form as the Euler-Lagrange equation (36) (a phenomenon already encountered in the case of the Lane-Emden equation for  $n = 1$ , recall Example 6.2.2). Thus, the general solution to the Jacobi equation is

$$f(x) = c_1 \exp(2\sqrt{\omega} \exp(x/2)) + c_2 \exp(-2\sqrt{\omega} \exp(x/2)),$$

with  $c_1, c_2$  constants of integration which can be fixed by the initial conditions  $f(0) = 0$ ,  $f'(0) = 1$ , giving

$$c_1 = \frac{e^{-2\sqrt{\omega}}}{2\sqrt{\omega}}, \quad c_2 = -\frac{e^{2\sqrt{\omega}}}{2\sqrt{\omega}}.$$

Substituting in the expression for  $f(x)$  above, the solution to the Jacobi equation can be written as

$$\begin{aligned} f(x) &= \frac{1}{2\sqrt{\omega}} \left( \exp((e^{x/2} - 1)2\sqrt{\omega}) - \exp(-(e^{x/2} - 1)2\sqrt{\omega}) \right) \\ &= \frac{1}{\sqrt{\omega}} \sinh(2\sqrt{\omega}(e^{x/2} - 1)), \end{aligned}$$

which clearly shows that there are no conjugate points to  $x = 0$ . The solutions (37) are thus minimals on their interval of definition.

**6.4. Square root Hamiltonian with a dissipation term.** Square-root Hamiltonians (or Lagrangians) are a standard feature of many reparametrization invariant field theories [11], [54]. The action of a relativistic particle [18], [22], [48], and the action of the Nambu string are familiar examples [4], [78]. Further examples of physical theories where this sort of Hamiltonians appear include general relativity [22], [50], as well as certain approaches to quantum gravity [12], [50], and also, they appear in brane motivated models [4], [58]. Aspects on the quantization of these kind of Hamiltonian theories may be found in [6] and [54], to mention some. In this section, we will study the Hamiltonian for a free particle under relativistic motion with a linear dissipation term, as proposed in [31]. This Hamiltonian reads

$$H(p, x, t) := e^{\gamma t} \sqrt{1 + p^2 e^{-2\gamma t}}, \quad (38)$$

where  $p$  stands for the canonical momentum associated to the dependent variable  $x = x(t)$ , and  $\gamma$  is the dissipation term. As discussed in [31], in the low velocity regime, this Hamiltonian reduces to the Caldirola-Kanai Hamiltonian which describes the motion of a non-relativistic particle with a linear dissipation term [57]. The Lagrangian associated to (38) is given by

$$L(t, x, \dot{x}) = -e^{\gamma t} \sqrt{1 - (\dot{x})^2}, \quad (39)$$

and thus the solution to Euler-Lagrange equation

$$\frac{d}{dt} \left( \frac{e^{\gamma t} \dot{x}}{\sqrt{1 - (\dot{x})^2}} \right) = 0 \quad (40)$$

is given by

$$x(t) = -\frac{A_0 \operatorname{arcsinh}(e^{-\gamma t} |A_0|)}{\gamma |A_0|}, \quad (41)$$

where  $A_0$  is an integration constant. The functions (7) and (8) are  $P = e^{\gamma t}/(1 - (\dot{x})^2)^{3/2}$  and  $Q = 0$ , respectively, and thus the Jacobi equation yields

$$\frac{d}{dt} \left( \frac{e^{\gamma t}}{(1 - (\dot{x})^2)^{3/2}} \frac{df}{dt} \right) = 0. \quad (42)$$

In this last equation, the term  $\dot{x}$  must be understood as the time-depending function  $\dot{x}(t) = A_0 e^{-\gamma t} / \sqrt{1 + A_0^2 e^{-2\gamma t}}$ . Therefore, for this model we are able to explicitly find the solution to Jacobi equation (42)

$$f(t) = c_0 - \frac{c_1}{\gamma} \frac{e^{-\gamma t}}{\sqrt{1 + A_0^2 e^{-2\gamma t}}}, \quad (43)$$

being  $c_0$  and  $c_1$  integration constants. We then note that this solution has a unique zero at the value  $t = (1/\gamma) \log \left( \sqrt{c_1^2 - A_0^2 \gamma^2 c_0^2} / \gamma c_0 \right)$ , so there are no conjugate points for the function  $f(t)$ , and the solution (41) is a minimum for the action on any interval  $[0, b]$ ,  $b \in \mathbb{R}$ .

**6.5. Quartic potential model.** In this section we will develop a model inspired by the static kink of the well-known  $\phi^4$  model in quantum field theory [55], [42]. The model can be resolved both on classical and quantum grounds, and contains soliton solutions (see below). In the context of brane theories, the so-called kink model also appears by the inclusion of rigidity terms associated to the intrinsic

curvature in their effective actions [77], [52].

The Lagrangian for our model is

$$L(t, x, \dot{x}) = \frac{1}{2}(\dot{x})^2 + \frac{\lambda}{4} \left( x^2 - \frac{m^2}{\lambda} \right)^2, \quad (44)$$

where  $m$  and  $\lambda > 0$  are arbitrary real constants. The Euler-Lagrange equation associated to this Lagrangian reads

$$\ddot{x} - \lambda x \left( x^2 - \frac{m^2}{\lambda} \right) = 0. \quad (45)$$

This equation is of Lénard type:  $\ddot{x} + f(x)\dot{x} + g(x) = 0$ , where  $f \equiv 0$  and  $g(x) = -\lambda x \left( x^2 - \frac{m^2}{\lambda} \right)$ . The change of variables  $u(x) = \dot{x}$  converts it in the first-order equation  $uu' = \lambda x \left( x^2 - \frac{m^2}{\lambda} \right)$ , which is immediately integrated to give

$$\int \frac{dx}{\sqrt{\lambda x^4 - 2m^2 x^2 - 2b}} = \frac{1}{\sqrt{2}} \int dt = \frac{t-a}{\sqrt{2}}, \quad (46)$$

where  $a, b \in \mathbb{R}$  are integration constants. The solutions commonly found in the literature (cited above) are obtained by taking  $b = -\frac{m^4}{2\lambda}$ , so to get a perfect square in the radicand of (46). In this way, the resulting solutions are:

$$y(t) = \pm \frac{m}{\sqrt{\lambda}} \tanh \left( \frac{m(t-a)}{\sqrt{2}} \right).$$

The solution with the plus sign is commonly termed the kink solution, while the one with the minus sign is called the anti-kink solution. Both solutions are bounded by the values  $\pm m/\sqrt{\lambda}$ . In particular, the energy density of the kink solution goes as the fourth power of the hyperbolic secant, and is localised within a width characterised by the quantity  $l/m$  [55]. However, other solutions exist. For instance, we could as well take  $b = 0$  in (46) to get (through an obvious change of variable):

$$\frac{t-a}{\sqrt{2}} = \int \frac{dx}{\sqrt{\lambda x^4 - 2m^2 x^2}} = \frac{1}{m\sqrt{2}} \int \frac{d\eta}{\eta\sqrt{\eta^2 - 1}} = \frac{1}{m\sqrt{2}} \arcsin \eta,$$

and hence the solution to the Euler-Lagrange equation (45):

$$x(t) = m\sqrt{\frac{2}{\lambda}} \sec(m(t-a)). \quad (47)$$

In order to get the Jacobi equation (12), we consider the functions  $P = 1 > 0$  and  $Q = 3\lambda x^2 - m^2$ , which set the equation for the function  $f(t)$ :

$$\frac{d^2 f}{dt^2} + m^2 (1 - 6 \sec^2(m(t-a))) f = 0. \quad (48)$$

We can take  $a = 0$  and  $m = 1$  without loss of generality (these are just re-scalings). Then, the Jacobi equation has the form  $\ddot{f} + \phi(t)f = 0$ , where  $\phi(t) = 1 - \sec^2 t \leq 0$  in the interval  $] -\pi/2, \pi/2[$ . At the points  $t = \pm\pi/2$ , the solutions have a blow-up and are not defined (so they can not be extended beyond these points). Thus, the solutions to the Jacobi equation are defined on  $] -\pi/2, \pi/2[$ , do not possess conjugate points in this interval (see Theorem 5.4) and the solution (48) is a true minimum on  $] -\pi/2, \pi/2[$ .

**6.6. Probability density and maximal entropy.** In this section we implement a constrained Lagrangian system related to a probability density function [64]. In Bayesian probability theory and in statistical mechanics, this system is related to the principle of maximum entropy [40], which also appears in other branches of physics, and in chemistry and biology [8], [43], [49], [71]. The model is defined as follows. Let  $Z$  be a random variable, and let  $\rho(x)$  its associated density function, so  $\rho : \mathbb{R} \rightarrow ]0, +\infty[$ . Suppose that we know the second order momentum

$$\sigma^2 = \int_{\mathbb{R}} x^2 \rho(x) dx, \quad (49)$$

and that we want to obtain the least biased density function  $\rho(x)$ . This may be written as the problem of finding the maximals for the entropy functional (defined in terms of the information theory):

$$S(\rho) = - \int_{\mathbb{R}} \rho(x) \log \rho(x) dx, \quad (50)$$

and subject to the constraints

$$\int_{\mathbb{R}} \rho(x) dx = 1, \quad (51)$$

$$\int_{\mathbb{R}} x^2 \rho(x) dx = \sigma^2. \quad (52)$$

Notice that the Lagrangian here, is defined on  $U = \mathbb{R} \times ]0, +\infty[ \times \mathbb{R}$ , although its dependence on the first and third variables is trivial.

Thus, we have the three functionals (in the notation of subsection 2.3)

$$\begin{aligned} S(\rho) &= - \int_{\mathbb{R}} \rho(x) \log \rho(x) dx, \\ K_1(\rho) &= \int_{\mathbb{R}} \rho(x) dx, \\ K_2(\rho) &= \int_{\mathbb{R}} x^2 \rho(x) dx. \end{aligned}$$

It is immediate to compute the variations:

$$\begin{aligned} \delta S(\rho, h) &= - \int_{\mathbb{R}} h(x) (1 + \log \rho(x)) dx, \\ \delta K_1(\rho, h) &= \int_{\mathbb{R}} h(x) dx, \\ \delta K_2(\rho, h) &= \int_{\mathbb{R}} x^2 h(x) dx, \end{aligned}$$

so it is obvious that they are weakly continuous. Let us apply the theorem 2.15 on Lagrange multipliers. The case  $\det(\delta K_i(y, h_l)) = 0$  ( $1 \leq i, l \leq 2$ ), would lead to

$$\det \begin{pmatrix} \int_{\mathbb{R}} h_1(x) dx & \int_{\mathbb{R}} h_2(x) dx \\ \int_{\mathbb{R}} x^2 h_1(x) dx & \int_{\mathbb{R}} x^2 h_2(x) dx \end{pmatrix} = 0$$

for arbitrary  $h_1, h_2 \in X$ , or:

$$\frac{\int_{\mathbb{R}} x^2 h_1(x) dx}{\int_{\mathbb{R}} h_1(x) dx} = \frac{\int_{\mathbb{R}} x^2 h_2(x) dx}{\int_{\mathbb{R}} h_2(x) dx},$$

which is absurd. Thus, we can introduce two Lagrange multipliers  $\lambda_1, \lambda_2$  and consider the Lagrangian

$$L(x, \rho, \rho') = -\rho(x) \log \rho(x) + \lambda_1 \rho(x) + \lambda_2 x^2 \rho(x).$$

The Euler-Lagrange equation yields

$$-\log \rho(x) - 1 + \lambda_1 + \lambda_2 x^2 = 0$$

with solution  $\rho(x) = e^{-1+\lambda_1+\lambda_2 x^2}$ . Substitution of this solution into the constraints (51) and (52) implies that the Lagrange multipliers are equal to  $\lambda_1 = 1 + \log \frac{1}{\sqrt{2\pi}\sigma}$  and  $\lambda_2 = -1/(2\sigma^2)$ , respectively. Thus, the solution of the Euler-Lagrange equation reads

$$\rho(x) = (\sqrt{2\pi}\sigma)^{-1} \exp(-x^2/(2\sigma^2)). \quad (53)$$

Finally, we see that the original Lagrangian  $L_0(x, \rho, \rho') = -\rho(x) \log \rho(x)$ , can actually be seen as a real function of a single variable on  $]0, +\infty[$ , for which the second derivative  $L_0''(\rho) = -1/\rho$  is always negative. Therefore,  $L_0$  is concave and, due to Theorem 4.3 and Remark 4.4, the solution obtained is a (global) maximal in  $]0, +\infty[$ .

**Remark 6.1.** *A posteriori, we see that the solution we have found, (53), belongs to  $\mathcal{C}^1(\mathbb{R})$ . However, this is not obvious a priori. Indeed, the method of Lagrange multipliers is not the best one to deal with the problem involving higher order moments, precisely because the eventual solution may lie outside the space from which we start, see [37].*

**Acknowledgements:** The research oh the third author (JAV) was partially supported by the Mexican Consejo Nacional de Ciencia y Tecnología CONACyT Project CB-2012 179115.

## REFERENCES

- [1] R. A. Adams, *Sobolev spaces* (Academic Press, 1978).
- [2] R. P. Agarwal and D. O'Regan, *An introduction to ordinary differential equations* (Springer Verlag, 2008).
- [3] J. C. Baez, *Spin foam models*, Class. Quant. Grav. **15** (1998) 1827–1858.
- [4] L. M. Baker and D. B. Fairlie, *Hamilton-Jacobi equations and Brane associated Lagrangians*, Nucl. Phys. **B596** (2001) 348–364.
- [5] J. L. Basdevant, *Variational principles in Physics* (Springer, 2010).
- [6] V. Berezin, *Square-root quantization: application to quantum black holes*, Nucl. Phys. Proc. Suppl. **57** (1997) 181–183.
- [7] J. Binney and S. Tremain, *Galactic dynamics* (Princeton University Press, 1994).
- [8] B. Buck and V. A. Macaulay (eds), *Maximum Entropy in Action: A Collection of Expository Essays* (Oxford University Press, 1991).
- [9] D. N. Burghes and A. Graham, *Introduction to Control Theory, Including Optimal Control* (Wiley, 1980).
- [10] A. Carlini, V. P. Frolov, M. B. Mensky, I. D. Novikov, and H. H. Soleng, *Time machines: the Principle of Self-Consistency as a consequence of the Principle of Minimal Action* Int. J. Mod. Phys. **D4** (1995) 557–580.
- [11] A. Carlini, and J. Greensite *Square Root Actions, Metric Signature, and the Path-Integral of Quantum Gravity*, Phys. Rev. **D52** (1995) 6947–6964.
- [12] S. Carlip, *(2 + 1)-Dimensional Chern-Simons Gravity as a Dirac Square Root*, Phys. Rev. **D45** (1992) 3584–3590; Erratum-ibid **D47** (1993) 1729.
- [13] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (Dover publications, 1967).

- [14] C. V. Coffman and J. S. V. Wong, *Oscillation and nonoscillation of solutions of generalized Emden-Fowler equations*, Transactions of the AMS **167** (1972) 399–434.
- [15] G. W. Collins, *The fundamentals of stellar astrophysics* (Freeman, 1989).
- [16] R. F. Curtain and A. J. Pritchard, *Functional analysis in modern applied mathematics* (Academic Press, 1977).
- [17] R. Emden, *Gaskugeln, Anwendungen der mechanischen Warmen-theorie auf Kosmologie und meteorologische Probleme* (B. G. Teubner, 1907).
- [18] P. P. Fiziev, *Relativistic Hamiltonian with square root in the path integral formalism*, Theor. Math. Phys. **62** 2, (1985) 123–130.
- [19] T. M. Flett, *Differential analysis* (Cambridge UP, 1980).
- [20] R. H. Fowler, *The form near infinity of real, continuous solutions of a certain differential equation of the second order*, Quart. J. Math. **45** (1914) 289–350; *Some results on the form near infinity of real continuous solutions of a certain type of second order differential equations*, Proc. London Math. Soc. **13** (1914) 341–371; *The solution of Emden’s and similar differential equations*, Monthly Notices Roy. Astro. Soc., **91** (1930) 63–91; *Further studies of Emden’s and similar differential equations*, Quart. J. Math. **2** (1931) 259–288.
- [21] C. Fox, *An introduction to the calculus of variations*, Cambridge UP (1963), reprinted by Dover (1987).
- [22] J. L. Friedman, J. Louko and S. N. Winters-Hilt, *Reduced phase space formalism for spherically symmetric geometry with a massive dust shell*, Phys. Rev. **D56** (1997) 7674–7691.
- [23] I. M. Gelfand and S. V. Fomin, *Calculus of variations* (Dover, 2000).
- [24] P. L. García, *The Poincaré-Cartan invariant in the calculus of variations*, in Symposia Mathematica XIV, pp. 5571–610, (Academic Press, 1974).
- [25] B. C. Garrett, N. Abusalbi, D. J. Kouri and D. G. Truhlar, *Test of variational transition state theory and the least action approximation for multidimensional tunneling probabilities against accurate quantal rate constants for a collinear reaction involving tunneling into an excited state*, J. Chem. Phys. **83** (1985) 2252.
- [26] G. Giachetta, L. Mangiarotti and G. Sardanashvily, *Advanced classical field theory*, (World Scientific, 2009).
- [27] M. Giaquinta and S. Hildebrandt, *Calculus of variations I: The Lagrangian formalism* (Springer Verlag, 1996).
- [28] H. Goenner and P. Havas, *Exact solutions of the generalized Lane–Emden equation*, J. Math. Phys. **41** 10, (2000) 7029–7043.
- [29] H. Goldschmidt and S. Sternberg, *The Hamilton-Cartan formalism in the calculus of variations*, Ann. Inst. Fourier (Grenoble) **23** (1973) 203–267.
- [30] H. Goldstein, C. Poole and J. Safko, *Classical Mechanics*, 3rd. ed. (Addison-Wesley, 2001).
- [31] G. González, *Relativistic motion with linear dissipation*, Int. J. Theor. Phys. **46** 3, (2007) 417–423.
- [32] C. G. Gray, G. Karl, and V. A. Novikov, *Progress in classical and quantum variational principles*, Rep. Prog. Phys. **67** (2004) 159–208.
- [33] C. G. Gray and E. Poisson, *When action is not least for orbits in general relativity*, Am. J. Phys. **79** 1, (2011) 43–56.
- [34] C. G. Gray and E. F. Taylor, *When action is not least*, Am. J. Phys. **75** 5, (2007) 434–458.
- [35] H. W. Hamber and R. M. Williams, *Discrete gravity in one dimension*, Nuclear Physics **B451** (1995) 305–324.
- [36] L. N. Hand and J. D. Finch, *Analytical Mechanics* (Cambridge University Press, 1998).
- [37] P. Harremoës and F. Topsøe, *Maximum Entropy Fundamentals*, Entropy **3** (2001) 191–226.
- [38] H. Hermes and J. P. Lasalle, *Functional analysis and time optimal control* (Academic Press, 1969).
- [39] G. P. Horedt, *Polytropes: Applications in astrophysics and related fields*, (Kluwer, 2004).
- [40] E. T. Jaynes, *Information theory and statistical mechanics*, Phys. Rev. **106** (1957) 620–630.
- [41] J. V. José and E. J. Saletan, *Classical Dynamics, a contemporary approach* (Cambridge University Press, 1998).
- [42] A. A. Kapustnikov, A. Pashnev and A. Pichugin, *The canonical quantization of the kink-model beyond the static solution*, Phys. Rev. **D55** (1997) 2257–2264.
- [43] J. F. Klein, *Physical significance of entropy or of the second law* (Cornell University Library, 2009).

- [44] O. Krupková, *The geometry of ordinary variational equations*, Lecture Notes in Mathematics **1678**, (Springer Verlag, 1997).
- [45] O. Krupková and D. J. Saunders (eds.), *Variations, geometry and physics*, (Nova Science Publishers, 2009).
- [46] I. J. H. Lane, *On the theoretical temperature of the sun under the hypothesis of a gaseous mass maintaining its volume by its internal heat and depending on the laws of gases known to terrestrial experiment*, Amer. J. Sci. and Arts, **4** (1870) 57–74.
- [47] L. P. Lebedev and M. J. Cloud, *The calculus of variations and functional analysis (with optimal control and applications in mechanics)* (World Scientific, 2003).
- [48] W. Lucha and F. F. Schöberl, *Relativistic Coulomb Problem: Energy Levels at the Critical Coupling Constant Analytically*, Phys. Lett. **B387** (1996) 573–576.
- [49] L. M. Martyushev and V. D. Seleznev, *Maximum entropy production principle in physics, chemistry and biology*, Physics Reports **426** (2006) 1–45.
- [50] P. Menotti, *Hamiltonian structure of 2+1 dimensional gravity*, In: Recent developments in general relativity, 14th SIGRAV Conference on General Relativity and Gravitational Physics, Genova, Italy (2000). Edited by R. Cianci, R. Collina, M. Francaviglia, P. Fré. (Springer, 2002).
- [51] T. A. Moore, *Getting the Most Action from the Least Action: A proposal*, Am. J. Phys. **72** 4, (2004) 522–527.
- [52] A. P. Nersisyan, *Hamiltonian formalism for particles with a generalized rigidity*, Theor. Math. Phys. **117** No. 1, (1998) 1214–1222.
- [53] L. A. Pars, *An introduction to the calculus of variations*, Heinemann (1962), reprinted by Dover (2010).
- [54] R. Puzio, *On the square root of the Laplace–Beltrami operator as a Hamiltonian* Class. Quantum Grav. **11** (1994) 609.
- [55] R. Rajaraman, *Solitons and Instantons* (North–Holland Publishing, 1988).
- [56] P. Ramond, *Field theory: A modern primer (Frontiers in Physics series Vol. 74)* (Westview Press, 2001).
- [57] M. Razavy, *Classical And Quantum Dissipative Systems* (Imperial College Press, 2006).
- [58] E. Rojas, *Higher order curvature terms in Born-Infeld type brane theories*, Int. J. Mod. Phys. **D20** (2011) 59–75.
- [59] D. J. Saunders, *An alternative approach to the Cartan form in Lagrangian field theories*, J. Phys. A **20** (1987) 339–349.
- [60] H. Sagan, *Introduction to the calculus of variations* (Dover, 1992).
- [61] G. F. Simmons and S. G. Krantz, *Differential Equations: Theory, Technique, and Practice* (McGraw-Hill, 2006).
- [62] D. Smith, *Variational methods in optimization* (Dover, 1998).
- [63] H. Stephani, D. Kramer, M. MacCallum, 426 C. Hoenselaers and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, 2003).
- [64] D. Stirzaker, *Elementary probability* (Cambridge University Press, 2003).
- [65] H. J. Sussmann and J. C. Willems, *300 years of optimal control: from the brachistochrone problem to the maximum principle*, IEEE Control Systems **17** (1997) 32–44.
- [66] E. F. Taylor, *Guest Editorial: A Call to Action*, Am. J. Phys. **71** 5 (2003) 423–425.
- [67] J. R. Taylor, *Classical mechanics* (University Science Books, 2005).
- [68] S. T. Thornton and J. B. Marion, *Classical Dynamics of Particles and Systems* (Brooks/Cole, 2004).
- [69] J. L. Troutman, *Variational Calculus and Optimal Control: Optimization With Elementary Convexity* (Springer Verlag, 1996).
- [70] B. Van Brunt, *The calculus of variations* (Springer Verlag, 2004).
- [71] Q. A. Wang, *Maximum entropy change and least action principle for non equilibrium systems*, Astrophysics and Space Sciences **305** (2006) 273.
- [72] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, 4th. ed (Dover, 1937).
- [73] R. M. Williams, *Recent Progress in Regge Calculus*, Nucl. Phys. Proc. Suppl. **57** (1997) 73–81.
- [74] J. S. W. Wong, *On the generalized Emden-Fowler equation*, SIAM Rev. **17** 2(1975) 339–360.
- [75] A. J. Zaslavski, *Turnpike properties in the calculus of variations and optimal control* (Springer Verlag, 2006).



- [76] E. Zeidler, *Nonlinear functional analysis and its applications, Vol. III: Variational methods and optimization* (Springer Verlag, 1986).
- [77] K. G. Zloshchastiev, *Quantum kink model and  $SU(2)$  symmetry: Spin interpretation and  $T$ -violation*, J. Phys. A: Math. Gen. **31** (1998) 6081–6085.
- [78] B. Zwiebach, *A First Course in String Theory* (Cambridge University Press, 2004).

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